Anomalous behavior of the contact process with aging

S. N. Dorogovtsev [1,2,*] and J. F. F. Mendes [1,†]

[1]*Departamento de Física and Centro de Física do Porto, Faculdade de Ciências, Universidade do Porto,*
Rua do Campo Alegre 687, 4169-007 Porto, Portugal
[2]*A. F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia*

The effect of power-law aging on a contact process is studied by simulation and using a mean-field approach. The introduced type of aging accounts for, e.g., the growth of the virus fitness (HIV infection). We find that the system may approach its stationary state in a nontrivial, nonmonotonous way. For the particular value of the aging exponent $\alpha = 1$ we observe a rich set of behaviors: depending on the process parameters, the relaxation to the stationary state proceeds as $1/\ln t$ or via a power law with a nonuniversal exponent. Simulation results suggest that for $0 < \alpha < 1$, the absorbing-state phase transition is in the universality class of directed percolation.

The study of systems with absorbing-state phase transitions has been a topic of intense research in recent years [1–3]. Among these systems, the contact process (CP), introduced by Harris [4] as a toy model for spreading of disease, has become the prototype model [5–9]. In fact, it is a dynamical version of directed percolation [3,10–16]. One should note that effects of aging, in the particular case of spreading of diseases were considered by Bernoulli [17] a long time ago. Nevertheless, importance of one factor was realized only recently. It is clear now that because of a fast mutation rate a virus becomes more resistant. The most striking example of such behavior is demonstrated by the HIV infection due to rapid replication of this virus [18–21]. In this respect, an ''old guy'' is ''stronger'' than a young one. [Of course, it is impossible to ascribe age directly to a rapidly replicated virus but it is quite reasonable to speak about the time passed after the instant when a particular cell was infected and a new virus lineage (strain) appeared [21]. All this time the infection is under permanent attacks of the immune system and drugs.] Sociology—evolution of *vox populi*—provides another example of a similar effect. After a person becomes set in his ways, nothing may change his beliefs. Therefore, aged opinions are more survivable than new ones. In our paper, we study the effect of the mentioned factor on the simple contact process with a single absorbing state, i.e., we consider the simplest idealized situation.

Recently, much effort has been directed toward generalizing the original models of absorbing-state phase transitions by introducing many absorbing states, and toward finding paths to self-organized critical phenomena [22–24]. Therefore, one can hardly expect that an ordinary CP will reveal striking features. Nevertheless, in the present paper, we demonstrate that even the simple CP with a single absorbing state shows anomalous behavior if one introduces such aging into the model.

We show that the system with power-law aging, before it approaches its stationary state, behaves nonmonotonously:

the time derivative of the particle density may change sign during the evolution. The case of the aging exponent α equal to one is especially interesting. In this marginal situation, in a mean-field approach, we find different types of behavior. Depending on the parameters of the system, the density of particles changes at long times according to a power law with nonuniversal exponents, or proportional to $1/\ln t$. Such slow relaxation is certainly not typical of absorbing-state phase transitions with a unique absorbing state. At their critical point, relaxation normally follows a power law [2,3].

We simulate the aging contact process (ACP) using sequential dynamics. Each site of the one-dimensional lattice may be empty or filled by one particle. At each unit of time a particle is chosen at random. We decide to annihilate this particle (a) or create a new one (b) with probabilities $P_a(s) = 1/[1 + p(s+1)^\alpha]$ or $P_c(s) = p(s+1)^\alpha/[1 + p(s+1)^\alpha]$ correspondingly. Here, s is the age of the particle and α is the aging exponent. In the case of annihilation, (a), the particle is deleted and we proceed to the next step increasing time by one unit. In case (b), we chose one of two nearest-neighbor sites with equal probability. If this site is filled, creation is impossible, and we increase time and proceed to the next step. If it is empty, we create a new particle at this site, increase time, and pass to the next step. Thus, as compared with the ordinary CP, we introduce the age-dependent probabilities $P_a(s)$ and $P_c(s)$. For $\alpha > 0$, the case studied here, $P_a(s)$ tends to zero and $P_c(s)$ approaches 1 at long times. According to the introduced rules, an ''old guy'' is certainly ''stronger'' than a young one.

The process is started with a random configuration of particles. Simulation times are about 10^8 Monte Carlo steps. The lattice size is taken up to 10^6 sites, to minimize the fluctuations of the particle density $n(t)$, which is the main quantity of interest. The simulation results are presented in Figs. 1–3. First, one sees that at each α value studied, $n(t)$ may behave nonmonotonically, depending on the initial conditions and parameters of the process. If, for instance, the stationary state is $n(\infty) = 1$ (see Figs. 2 and 3), the density may first decay nearly to zero, stay in this range for a long time, and only approach the $n(\infty) = 1$ much later. Frequently, it is only with long-time simulations that we can

*Electronic address: sdorogov@fc.up.pt
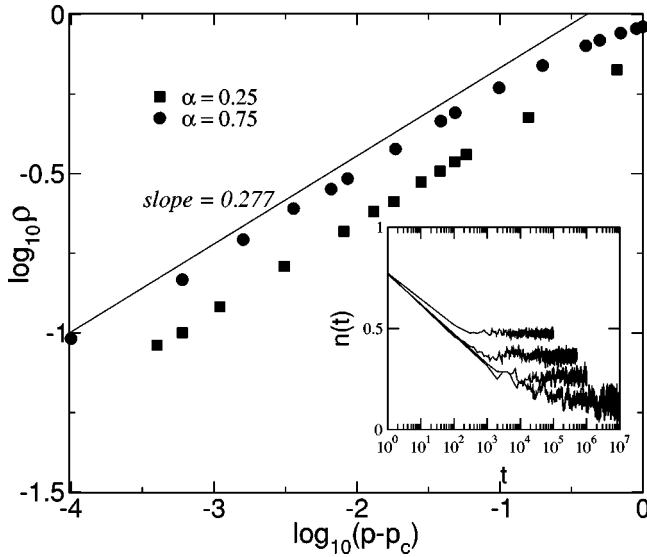†Electronic address: jfmendes@fc.up.pt

FIG. 1. Log-log plot of the stationary density ρ vs $p-p_c$ for two different values of the aging exponent α, 0.25 and 0.75. The critical points are $p_c(0.25)=2.3419(3)$ and $p_c(0.75)=1.1014(3)$. The full line has a slope corresponding to the β exponent of DP in the $1+1$ dimension. The inset shows the evolution of $n(t)$ for different values of p (from top to below, $p=2.5$, 2.4, 2.36, and 2.3423) and $\alpha=0.25$.

observe the final stage of the relaxation. Such behavior resembles a typical course of the HIV infection: after the primary infection peak, the level of infected cells stays low for many years until the final development of the disease.

Second, in the case of $0<\alpha<1$, the critical characteristics are the same as for the contact process without aging, i.e., $\alpha=0$. In Fig. 1, we show the stationary-particle density as a function of the deviation from the critical point p_c for two different values of the aging exponent, $\alpha=0.25$ and 0.75. In both cases the data suggest that the critical exponent
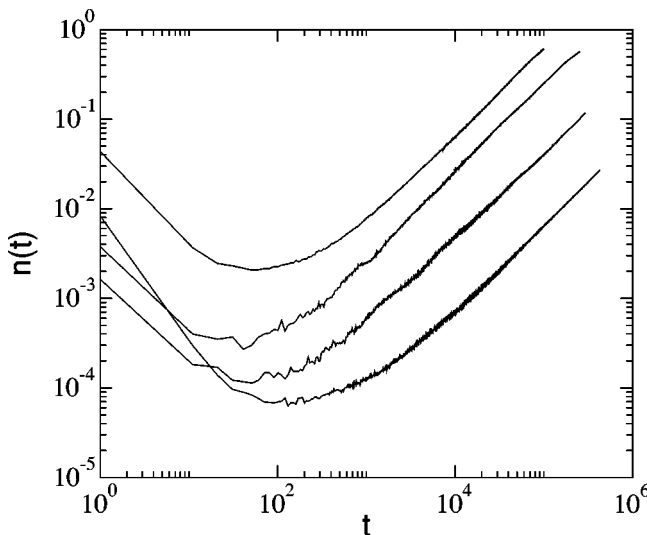

FIG. 2. Evolution of $n(t)$ for different values of p and initial densities, for $\alpha=2$. From top to below, $p=0.075$, $n_0=0.01$; $p=0.075$, $n_0=0.25$; $p=0.05$, $n_0=0.05$; and $p=0.1$, $n_0=0.01$.
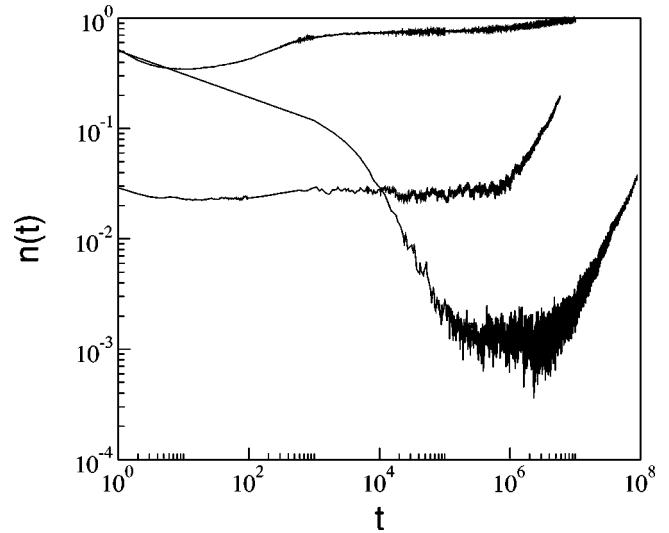

FIG. 3. Temporal evolution of the particle density $n(t)$ for three different values of p and $\alpha=1$. From top to below, $p=0.65$, 0.6667, and 0.75.

β is not α dependent, and agrees with the directed percolation value, $\beta\simeq0.277$. When $\alpha>1$, the phase transition is absent and the only possible stationary state is $n(\infty)=1$; after a linear growth, the system approaches this value (see Fig. 2).

In the marginal case $\alpha=1$, we observe a complicated, slow evolution with several temporal regimes characterized by different behaviors. Due to the slow relaxation, it is difficult to fix the final state. In Fig. 3 we present the results for the temporal evolution of the particle density for several values of the parameter p. Depending on p, different types of the evolution are observed.

Let us present the simplest possible mean-field description of the ACP. While the model is even simpler than that used in our simulations, it is reasonable to expect that it can explain the principal features observed in the latter. As in the ordinary CP, each site of a lattice may be filled by one particle or be empty. Each particle has its age, s. At each increment of time, all lattice sites are updated according to the following rules. (i) The probability for a particle of age s to die at the next instant is p_s, irrespective of its environment. (ii) The probability for a particle to survive until the next update is therefore $1-p_s$. (iii) The probability that a particle will be born at an empty site in the next instant is $(1/n_S)\Sigma_{i=1}^{n_F}q_{s(i)}$, where n_S is the total number of nearest-neighbor sites, and n_F is the total number of particles at such sites. $s(i)$ is the age of the particle at site i. (iv) The probability for an empty site to remain empty until the next update is therefore $1-(1/n_S)\Sigma_{i=1}^{n_F}q_{s(i)}$. In this way we present a natural generalization of the CP, introducing age-dependent death and birth probabilities, p_s and q_s, respectively.

Let us derive the mean-field equations, introducing the following quantities: the total number of particles at time t, $n_t=\Sigma_{s=0}^{t}a_{t,s}$, where $a_{t,s}$ is the number of particles of the age s at time t. The initial condition is $a_{0,0}=n(0)$, where $n(0)$ is the initial number of particles.

Since the only possibility for an old particle ($s>0$) to survive is process (ii), the first equation is $a_{t+1,s+1}=(1-p_s)a_{t,s}$, i.e., a fraction $(1-p_s)$ of the particles with age s survive. One may verify the solution $a_{t,s}=a_{t-s,0}\Pi_{u=0}^{s-1}(1-p_u)$, where $\Pi_{u=0}^{-1}\equiv1$ by definition.

We demonstrate the derivation of the second equation in the simplest case of only two nearest-neighboring sites (one may show that other coordination numbers lead to the same equation). Particles are only created via process (iii). Hence,

$$a_{t+1,0}=(1-n_t)\sum_{s,u=0}^{t}a_{t,s}a_{t,u}\frac{q_s+q_u}{2}+2(1-n_t)^2\sum_{s=0}^{t}a_{t,s}\frac{q_s}{2}.$$
(1)

Thus,

$$a_{t+1,0}=(1-n_t)\sum_{s=0}^{t}a_{t,s}q_s.$$
(2)

The previous equations describe completely the ACP in the mean-field approach.

Applying $\Sigma_{s=0}^{t}$ to our first mean-field equation one obtains $n_{t+1}-n_t=a_{t+1,0}-\Sigma_{s=0}^{t}p_sa_{t,s}$. Taking Eq. (2) into account, we immediately obtain the equation generalizing the usual mean-field equation for the CP without aging,

$$n_{t+1}-n_t=\sum_{s=0}^{t}a_{t,s}[(1-n_t)q_s-p_s].$$
(3)

Now we see that all quantities may be expressed through the $a_{t,0}$, i.e., through the number of particles born at instant t. The equation for this variable follows from Eq. (2),

$$a_{t+1,0}=\left[1-\sum_{s=0}^{t}a_{t-s,0}\prod_{u=0}^{s-1}(1-p_u)\right]$$

$$\times\sum_{s=0}^{t}a_{t-s,0}q_s\prod_{u=0}^{s-1}(1-p_u).$$
(4)

Formally speaking, Eq. (4) together with the initial condition, define the solution of our problem. But the problem is still too difficult for analytical treatment. Here we consider only the simplest case of $q=$const and an s-dependent p_s. Let us pass to the continuous limit. Inserting the expression for $a_{t,s}$ into the definition of n_t, we get the following closed-integral equation for the total number of particles.

$$n(t)=n(0)\exp\left[-\int_0^t ds\, p(s)\right]+q\int_0^t ds\, n(s)[1-n(s)]$$

$$\times\exp\left[-\int_0^{t-s}du\, p(u)\right].$$
(5)

One may rewrite Eq. (3) in the following form: $n_{t+1}-n_t=(q-p_t)n_t-qn_t^2+\Sigma_{s=0}^{t}(p_t-p_s)a_{t,s}$. Inserting the expression for $a_{t,s}$ and passing to the continuous limit, one obtains

$$\frac{\partial n(t)}{\partial t}=[q-p(t)]n(t)-qn^2(t)+q\int_0^t ds\, n(s)[1-n(s)]$$

$$\times[p(t)-p(t-s)]\exp\left[-\int_0^{t-s}du\, p(u)\right].$$
(6)

Equation (6) may be also obtained by differentiating Eq. (5). If we set $p=$const, Eq. (6) reduces to the usual equation [2,3] for the CP without aging. We write out the known results on linear relaxation for it for later comparison: if $p>q$, $n(\infty)=0$ and $n(t)\propto\exp[-(p-q)t]$; for $p<q$, $n(\infty)=1-p/q$ and $n(t)-n(\infty)\propto\exp[-(q-p)t]$; and at $p=q$, $n(t)\propto1/t$.

The reasonable dependences for $p(s)$ and $q(s)$, which admit comparison with our simulations, are the following: $p(s)$ decreases gradually as the particle age s increases, and $q(s)$ increases with s increasing or is constant. Here, we consider the annihilation probability $p(s)=c(s+t_0)^{-\alpha}$, where α is the aging exponent. The constant c plays the same role as the parameter p^{-1} in our simulations. One can check that, in the particular case of $q=0$, i.e., when creation of particles is absent, Eq. (5) is valid for any dimension, and exact. In this case, at $0<\alpha<1$, $n(t)$ approaches zero, $n(t)=n(0)\exp[t_0^{1-\alpha}/(1-\alpha)]\exp[-(t+t_0)^{1-\alpha}/(1-\alpha)]$, at $\alpha>1$, $n(t)$ quickly approaches a constant value $n(\infty)=n(0)\exp\{-ct_0^{\alpha-1}/(\alpha-1)\}$, and, for $\alpha=1$, $n(t)=n(0)t_0/(t+t_0)$.

Equation (5) may be easily solved by iteration. One may start, e.g., from $n^{(i)}(t)=n(0)$. The resulting solutions $n(t)$ are very similar to those obtained by numerical simulation; we do not present these curves here, but only describe the results of the direct analysis of Eq. (5).

First of all, expanding Eq. (5) for small t one finds that

$$\frac{\partial n(0)}{\partial t}=n(0)\{q[1-n(0)]-ct_0^{-\alpha}\}.$$
(7)

[One may check that this relation is also valid for an s-dependent $q(s)$ if one replaces q by $q(0)$.] Hence, $n(t)$ may increase or decrease from $n(0)$ at short times depending on a particular relation between the constants of the problem and on the initial condition. In principle, the sign of the derivative is independent of the value $n(\infty)$. That leads to the observed nonmonotonic behavior of $n(t)$.

The nonmonotonous behavior observed in some regimes can be intuitively understood regarding that in the very beginning the particles are by necessity young, so most of them easily die off, but later the survivors turn to be nearly immortal, and the lattice will be filled up with old-timers.

Let us now consider the behavior of $n(t)$ for different values of α at long times. [Note that all the following asymptotes may be obtained from the Laplace transform of Eq. (5) linearized near the corresponding stationary state.]

Regime I, $0<\alpha<1$. The situation is very similar to the CP without aging. There is a critical point $c^*(\alpha,q,t_0)$ for each value of α and q. At $c>c^*(\alpha,q,t_0)$, $n(\infty)=0$ and $n(t)$ has an exponential approach to the stationary state. At the critical point, $c=c^*(\alpha,q,t_0)$, $n(t)$ relaxes to zero by a

power law, $n(t) \propto 1/t$. At $c < c^*(\alpha, q, t_0)$, at long times, $n(t)$ approaches $n(\infty) > 0$ exponentially, $n(t) - n(\infty) \propto \exp\{-g(\alpha, c, q, t_0)t\}$. Here, $g(\alpha, c, q, t_0)$ is zero at the critical point, and also approaches zero when α approaches 1.

Regime II, $\alpha > 1$. For any c and q, $n(t)$ approaches 1 exponentially. The critical point (and scaling) are absent. The reasons are the following. At $\alpha > 1$, the kernel $\exp\{-\int_0^t du c(u+t_0)^{-\alpha}\}$ quickly decreases to a constant value, $\exp\{-ct_0^{\alpha-1}/(\alpha-1)\}$, as t grows, and one can substitute this constant into Eq. (5). Taking the derivative of Eq. (5) and linearizing the resulting equation near $n(\infty) = 1$ we get

$$1 - n(t) \propto \exp\{-q \exp[-ct_0^{-(\alpha-1)}/(\alpha-1)] t\}. \quad (8)$$

Regime III, $\alpha = 1$, the most intriguing case. Several types of critical behavior are realized for different values of c.

(a) For $c < 1$ and any q, $n(\infty) = 1$ and the long-time dependence is $1 - n(t) \propto t^{-(1-c)}$. Depending on the relation between c, q, and t_0 [see Eq. (7)], $n(t)$ exhibits this dependence immediately or, on the contrary, $n(t)$ first decreases, remains near zero for some time, and then very slowly approaches 1. The kernel $\exp\{-\int_0^t ds\, p(s)\} = [(t+t_0)/t_0]^{-c}$, so if we demand $n(t \to \infty) \to 1$ in Eq. (5), we immediately obtain such a form for the asymptote.

(b) For $c = 1$ and any q, $n(\infty) = 1$ and the long-time behavior is $1 - n(t) \propto 1/\ln t$.

(c) For $1 < c < 1 + qt_0$, $n(\infty) = 1 - (c-1)/(qt_0)$ and $n(t) - n(\infty) \propto t^{-c}$.

(d) At $c = 1 + qt_0$, $n(\infty) = 0$ and $n(t) \propto 1/t$. The kernel is a quickly decreasing function as compared with $n(t)$, so the reason for such behavior is the same as for the $1/t$ relaxation at the critical point at $0 \le \alpha < 1$.

(e) At $c \ge 1 + qt_0$ [note that the definition of $p(s)$ imposes the additional restriction on possible values of c and t_0: $c < t_0$], $n(\infty) = 0$ and $n(t) \propto t^{-c}$.

The exponents $1 - c$ and c are *nonuniversal* since c is simply a coefficient in the definition of the annihilation probability. Previously, nonuniversality was found in models with multiple absorbing states [5,6,9,23–25] and in the process of spreading in media with memory [26]. Nevertheless, the kind of nonuniversality observed here was not seen.

One should note that the results for *Regime III* were obtained only in the frame of mean-field theory. We did not study the role of fluctuations, and cannot estimate their possible effect [27]. Our simulations do not let us fix the final-stationary state (except for the case $n(\infty) = 1$), when the relaxation is slow, i.e., for $\alpha = 1$. Note also that our mean-field approach is based on parallel dynamics while a sequential one was used in the numerical simulations. This difference, however, seems to be not crucial for the CP with a single absorbing state.

In summary, we have studied the influence of power-law aging on the contact process with a single absorbing state. The introduced type of aging accounts for, e.g., the growth of the virus fitness. This is realized in the case of the HIV infection because of the fast mutation rate due to its rapid replication of the virus. Of course, we do not model a real course of the HIV infection but only consider consequences of the most important factor, the growth of the virus fitness. Both simulation and mean-field theory reveal the nonmonotonic character of the relaxation. In the marginal case of a unit-aging exponent, in the frame of the mean-field theory, we found that the relaxation of the particle density to the stationary state proceeds by a power-law with a nonuniversal exponent, or even slower, i.e., proportional to $1/\ln t$.

[1] T.M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).

[2] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, England, 1999).

[3] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).

[4] T. E. Harris, Ann. Prob. **2**, 969 (1974).

[5] M. A. Muñoz, G. Grinstein, R. Dickman and R. Livi, Phys. Rev. Lett. **76**, 451 (1996).

[6] I. Jensen and R. Dickman, Phys. Rev. E **48**, 1710 (1993).

[7] P. Grassberger and A. de la Torre, Ann. Phys. (N.Y.) **122**, 373 (1979).

[8] U. Alon, M. R. Evans, H. Hinrichsen, and D. Mukamel, Phys. Rev. Lett. **76**, 2746 (1996).

[9] R. Dickman, Phys. Rev. E **53**, 2223 (1996).

[10] J. Blease, J. Phys. C **10**, 923 (1977).

[11] J. W. Essam and K. De'Bell, J. Phys. A **14**, L459 (1981).

[12] W. Kinzel, Z. Phys. B: Condens. Matter **58**, 229 (1985).

[13] H. K. Janssen, Z. Phys. B: Condens. Matter **42**, 151 (1981).

[14] P. Grassberger, Z. Phys. B: Condens. Matter **47**, 365 (1982).

[15] P. Grassberger, J. Phys. A **22**, L1103 (1984).

[16] J. L. Cardy and R. L. Sugar, J. Phys. A **13**, L423 (1980).

[17] D. Bernoulli, Hist. Acad. R. Sci. (Paris) Mém. Math. Phys. Mém. **1760** , 1 (1766).

[18] M. A. Novak, R. M. Anderson, A. R. McLean, T. F. W. Wolfs, J. Goudsmit, and R. M. May, Science **254**, 963 (1991).

[19] R. A. Weiss, Science **272**, 1885 (1996).

[20] M. A. Nowak and A. J. McMichael, Sci. Am. (Int. Ed.) **273**, 42 (1995).

[21] R. M. Zorzenon dos Santos and S. Countinho, e-print cond-mat/0008081.

[22] R. Dickman, M. A. Muñoz, A. Vespignani, and S. Zapperi, Braz. J. Phys. **30**, 27 (2000).

[23] J. F. F. Mendes, R. Dickman, M. Henkel, and M. C. Marques, J. Phys. A **27**, 3019 (1994).

[24] M. A. Muñoz, G. Grinstein, and R. Dickman, J. Stat. Phys. **91**, 541 (1998).

[25] C. Lopez and M. A. Muñoz, Phys. Rev. E **56**, 4864 (1997).

[26] P. Grassberger, H. Chaté, and G. Rousseau, Phys. Rev. E **55**, 2488 (1997).

[27] The situation in which the considered type of aging (with $\alpha = 1$) effectively arised was simulated in R. Cafiero, A. Gabrielli, and M. A. Muñoz, Phys. Rev. E **57**, 5060 (1998). Introducing of quenched disorder into a contact process produced effective probabilities, that changed in time depending on what happened before, and continuously varying exponents.